Course in Perugia, summer 2021.

Complex Analysis

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Text book:
Real and Complex Analysis, Walter Rudin Third Edition. International edition 1987.

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Contents

1. Introduction

The course is an elementary introduction to complex analysis. The main result in the course is the Riemann Mapping Theorem. We will cover all details needed to prove this theorem. On the way, we will come across many basic properties of analytic functions all of which will be proved in complete detail.

The course will use the textbook written by Walter Rudin. For ease of reference we include the numbers of the Theorems as they are in the book of Rudin, and also page numbers from Rudin's book. However, there is no need to actually read anything from this book.

We start by covering some material from Chapter 10, Elementary Properties of holomorphic functions.

We introduce some notation. Let $\mathbb{R}$ denote the real numbers. The complex plane $\mathbb{C}$ is given as $\mathbb{C} = \{x+iy; x, y \in \mathbb{R}\}$ and i stands for an imaginary unit with $i^2 = -1$. The complex plane can be naturally identified with $\mathbb{R}^2$ and this gives $\mathbb{C}$ a natural topology.

We define the absolute value $|z| = |x+iy|$ by $|z| = \sqrt{x^2+y^2}$.

If $r > 0$ and a is a complex number, then $D(a;r) = \{z; |z-a| < r\}$ is the open circular disc with center a and radius r. These are then a basis for the topology on $\mathbb{C}$.

We denote by $\overline{D}(a;r)$ the closure of $D(a;r)$ and $D'(a;r) = \{z; 0 < |z-a| < r\}$ is the punctured disc with center at a and radius r.

We say that a (nonempty) open set Ω is connected if it is not possible to write $\Omega = \Omega_1 \cup \Omega_2$ where the Ω_j are disjoint nonempty open sets.

Region is defined on page 197, but there is no number attached.

Definition 1.1. By a region we shall mean a non empty connected open subset of the complex plane.

Page 197. Definition 10.2 Analytic function

Definition 1.2. Suppose f is a complex function defined in Ω. If $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, we denote this limit by $f'(z_0)$ and we call it the derivative of f at z_0.

This means that there is a complex number $f'(z_0)$ so that for every $\epsilon > 0$, there is a $\delta > 0$ so that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

for all $z \in D'(z_0;\delta)$.

If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is holomorphic (or analytic) in Ω. We denote by $H(\Omega)$ the set of all analytic functions on Ω.

The following is immediate from the definition:

Remark 1.3. Analytic functions are continuous.

The Chain Rule is on page 197. It is not given a name, but the proof is included.

Theorem 1.4. *If $f \in H(\Omega)$, if $f(\Omega) \subset \Omega_1$, if $g \in H(\Omega_1)$, and if $h = g \circ f$, then $h \in H(\Omega)$, and h' can be computed by the chain rule*

$$h'(z_0) = g'(f(z_0))f'(z_0) \qquad (z_0 \in \Omega). \tag{1.1}$$

Proof. Fix $z_0 \in \Omega$. We can write, when $z \neq z_0$, $\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + \epsilon(z)$ where $\epsilon(z) \to 0$ when $z \to z_0$. Hence $f(z) - f(z_0) = (f'(z_0) + \epsilon(z))(z - z_0)$ Similarly, if we let $w_0 = f(z_0)$, then $g(w) - g(w_0) = (g'(w_0) + \eta(w))(w - w_0)$. Note that these equalities also hold if $z = z_0$ and/or $w = w_0$ if we define $\epsilon(z_0) = \eta(w_0) = 0$.

Combining these two, we get that $h(z) - h(z_0) = g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \eta(f(z))(f(z) - f(z_0)) = (g'(f(z_0)) + \eta(f(z))(f'(z_0) + \epsilon(z))(z - z_0)$ Note that by continuity of f, $\eta(f(z)) \to 0$ if $z \to z_0$.

If $z \neq z_0$, we get
$\frac{g(f(z))-g(f(z_0))}{z-z_0} = (g'(f(z_0)) + \eta(f(z))(f'(z_0) + \epsilon(z)) \to g'(f(z_0))f'(z_0)$.

□

Exercise 1.5. ..

2. Power series and their derivatives

A power series

$$\sum_{n=1}^{\infty} c_n(z-a)^n \ (1)$$

is said to be convergent for a point z if the sequence $\sum_{n=1}^{m} c_n(z-a)^n$ converges to a complex number as $m \to \infty$. We denote this number by $\sum_{n=1}^{\infty} c_n(z-a)^n$.

We will use a few facts. First is that if the series converges for a value of z and $0 < r < |z-a|$, then the series converges uniformly on $\Delta(a,r)$. Moreover if the series diverges (i.e. does not converge) for some z and $|w-a| > |z-a|$, then the series diverges at w. The radius of convergence R is the largest value for which the series converges whenever $|z-a| < R$. The number R is given by the root test as $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$.

We say that a function $f : \Omega \to \mathbb{C}$ is representable by a power series in Ω if for every disc $\Delta(a; r) \subset \Omega$, there corresponds a series (1) which converges to $f(z)$ for every $z \in \Delta(a; r)$.

Theorem 10.6, page 198

Theorem 2.1. *If f is representable by a power series in Ω, then $f \in H(\Omega)$ and f' is also representable by a power series in Ω. In fact if*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \ (1)$$

for $z \in D(a, r)$, then for these z we also have

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1} \ (2)$$

Proof. We first observe that by the root test, we see that if (1) converges in $\Delta(a; r)$ then (2) also converges there. We can for simplicity assume that $a = 0$. We denote the sum (2) by $g(z)$. We show that f is differentiable in $\Delta(0, r)$ and that $f' = g$.

Pick a $w \in \Delta(0, r)$ and choose ρ so that $|w| < \rho < r$. If $z \neq w$, then we get that

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} c_n \left[\frac{z^n - w^n}{z - w} - n w^{n-1} \right] \ (3)$$

Note that there is no term for $n = 0$. For $n = 1$, note that by convention we set $nw^{n-1} = 1$. It follows that there is no term for $n = 1$. So we consider the terms for $n \geq 2$. We will use the following formula:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + \cdots + a b^{n-2} + b^{n-1})$$

We can hence write

$$\frac{z^n - w^n}{z - w} = z^{n-1} + z^{n-2} w + \cdots + w^{n-1} \ (4)$$

Hence

$$
\begin{aligned}
\frac{z^n - w^n}{z - w} - nw^{n-1} &= (z^{n-1} - w^{n-1}) + (z^{n-2}w - w^{n-1}) + \\
\cdots &+ (zw^{n-2} - w^{n-1}) + (w^{n-1} - w^{n-1}) \\
&= (z-w)(z^{n-2} + wz^{n-3} + \cdots + w^{n-2}) \\
&+ w(z^{n-2} - w^{n-2}) + \cdots \\
&+ w^{n-2}(z-w) \\
&= (z-w)(z^{n-2} + wz^{n-3} + \cdots + w^{n-2}) \\
&+ (z-w)(wz^{n-3} + w^2 z^{n-4} + \cdots + w^{n-2}) \\
&+ \cdots + (z-w)(w^{n-2}) \\
&= (z-w)(z^{n-2} + 2wz^{n-3} + 3w^2 z^{n-4} + \cdots + (n-1)w^{n-2})
\end{aligned}
$$

Hence if $|z| < \rho$, we get that

$$\left| \frac{z^n - w^n}{z - w} - nw^{n-1} \right| \leq |z - w| \frac{n(n-1)}{2} \rho^{n-2} \quad (5)$$

It follows that

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2} \quad (6)$$

Since $\rho < r$ we get that the right side converges.

This says that $f'(w) = g(w)$ and completes the proof.

□

Exercise 2.2. ..

3. The local Cauchy Theorem

We introduce notation from the section Integration over Paths, page 200-202.

A **curve** γ is a continuous function $\gamma : [\alpha, \beta] \to \mathbb{C}$. The range of γ is called γ^* and is then a subset of the complex plane. If $\gamma(\alpha) = \gamma(\beta)$, then γ is a **closed curve.**

If the curve γ in addition has a piecewise continuous derivative, then γ is a **path** and if the curve in addition is closed, then γ is a **closed path**.

Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path and $f : \gamma^* \to \mathbb{C}$ is a continuous function, then we define

$$\int_\gamma f(z) dz = \int_\alpha^\beta f(\gamma(t)) \gamma'(t) dt.$$

Theorem 10.12, Page 204

Theorem 3.1. *Suppose $F \in H(\Omega)$ and F' is continuous in Ω. Then*

$$\int_\gamma F'(z)dz = 0$$

for every closed path in Ω.

Proof.

$$\begin{aligned}\int_\gamma F'(z)dz &= \int_\alpha^\beta F'(\gamma(t))\gamma'(t)dt \\ &= \int_\alpha^\beta (dF(\gamma(t))/dt)dt \\ &= F(\gamma(\beta)) - F(\gamma(\alpha)) \\ &= 0\end{aligned}$$

□

This has a corollary:

Corollary 3.2. *(Corollary to Theorem 10.12) Since z^n is the derivative of $\frac{z^{n+1}}{n+1}$ for all integers $n \neq -1$, we have*

$$\int_\gamma z^n dz = 0$$

for every closed path γ for $n = 0, 1, 2, \ldots$, and for those closed paths for which $0 \notin \gamma^*$, n=-2,-3,-4,...

Exercise 3.3. ..

4. Cauchy's Theorem in a convex set

Theorem 10.13, Page 205. Cauchy's Theorem for a triangle

Theorem 4.1. *Suppose Δ is a closed triangle in a plane open set Ω, $p \in \Omega$, f is continuous on Δ, and $f \in H(\Omega \setminus \{p\})$. Then*

$$(1) \quad \int_{\partial\Delta} f(z)dz = 0.$$

Definition 4.2. Let $\{a, b, c\}$ be an ordered triplet of numbers. Let $\Delta = \Delta(a, b, c)$ be the triangle with vertices at a, b and c. (Δ is the smallest closed convex set which contains a, b and c), and define

$$\int_{\partial\Delta} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f,$$

for any f continuous on the boundary of Δ.

First we assume that $p \notin \Delta$. Let a, b and c be the vertices of Δ, let a', b' and c' be the midpoints of $[b, c]$, $[c, a]$ and $[a, b]$ respectively, and consider the four triangles Δ^j formed by the ordered triplets

$$\{a, c', b'\}, \{b, a', c'\}, \{c, b', a'\}, \{a', b', c'\}.$$

If J is the value of the integral (1) we get that

$$J = \sum_{j=1}^{j=4} \int_{\partial \Delta^j} f(z) dz.$$

The absolute value of at least one of the 4 integrals is at least $|J/4|$. Call the corresponding triangle Δ_1, repeat the argument with Δ_1 in place of Δ and so forth.

This generates a sequence of triangles Δ_n such that $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$, such that the length of $\partial \Delta_n$ is $2^{-n} L$, if L is the length of $\partial \Delta$, and such $|J| \leq 4^n |\int_{\partial \Delta_n} f(z) dz|$ for all n.

By compactness, there is a point z_0 which belongs to all the triangles. In particular, f is differentiable at z_0.

Let $\epsilon > 0$. There exists an $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|$$

whenever $|z - z_0| < r$, and there exists an n such that $|z - z_0| < r$ for all $z \in \Delta_n$. By the Corollary to Theorem 10.12 applied to constant and linear functions,

$$\int_{\partial \Delta_n} f(z) dz = \int_{\partial \Delta_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)] dz,$$

so that we get

$$\left| \int_{\partial \Delta_n} f(z) dz \right| \leq \epsilon (2^{-n} L)^2$$

which implies that $|J| \leq \epsilon L^2$. Hence $J = 0$ if $p \notin \Delta$.

Assume next that p is a vertex of Δ, say $p = a$. If a, b and c are on the same line, so the triangle is degenerate, then (1) is anyhow trivial. So assume not. Then choose points $x \in [a, b]$ and $y \in [a, c]$, both close to a and observe that the integral of f over $\partial \Delta$ is the sum of integrals of the triangles $\{a, x, y\}, \{x, b, y\}$ and $\{b, c, y\}$. The last two are 0, since these triangles do not contain p. Hence the integral over $\partial \Delta$ is the sum of the integrals over $[a, x], [x, y]$ and $[y, a]$, and since these intervals can be made arbitrarily short and f is bounded on Δ, we again obtain (1).

Finally, if p is an arbitrary point of Δ, apply the preceeding result to $\{a, b, p\}, \{b, c, p\}$ and $\{c, a, p\}$ to complete the proof.

Page 206, Theorem 10.14. Cauchy's Theorem

Theorem 4.3. *Suppose Ω is a convex open set. $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega \setminus \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence $\int_\gamma f(z)dz = 0$ for every closed path γ in Ω.*

The proof uses Theorems 10.12 and Theorem 10.13.

Proof. We follow the proof in Rudin, page 207. Fix $a \in \Omega$. Since Ω is convex, Ω contains the straight line interval from a to z for every $z \in \Omega$, so we can define

$$F(z) = \int_{[a,z]} f(\xi)d\xi; \ (z \in \Omega).$$

For any z and $z_0 \in \Omega$, the triangle with vertices at a, z_0 and z lies in Ω. Hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by Theorem 10.13 or theorem 4.1 here (Cauchy's Theorem for a triangle, page 205)

Fixing z_0, we thus obtain if $z \neq z_0$:

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\xi) - f(z_0))d\xi.$$

Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)| < \epsilon$ if $|\xi - z_0| < \delta$. Hence the absolute value of the right side is less than ϵ as soon as $|z - z_0| < \delta$. This proves that $F' = f$. In particular, $F \in H(\Omega)$. It follows now from Theorem 10.12 page 204 or Theorem 3.1 here that if γ is any closed path in Ω, then $\int_\gamma f(z)dz = \int_\gamma F'(z)dz = 0$. □

Exercise 4.4. ..

5. Cauchy's Formula and power series.

Next Theorem 10.15, Page 207:

Theorem 5.1. *Suppose $\overline{\Delta(a;r)} \subset \Omega$ and $f \in H(\Omega)$. Let $\gamma = \partial\Delta(a;r)$ traversed counterclockwise. If $z \in \Delta(a;r)$, then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

Proof. Fix z and define g on Ω by $g(\xi) = \frac{f(\xi) - f(z)}{\xi - z}$ for $\xi \in \Omega \setminus \{z\}$ while $g(z) = f'(z)$. Then g satisfies the hypotheses of Theorem 10.14. Hence

$$\frac{1}{2\pi i} \int_\gamma g(\xi)d\xi = 0.$$

So

$$\int_\gamma \frac{f(\xi)}{\xi - z} d\xi = f(z) \int_\gamma \frac{d\xi}{\xi - z}.$$

We need to show that $\int_\gamma \frac{d\xi}{\xi - z} = 2\pi i$ if $z \in \Delta(a;r)$. Dividing $\Delta(a;r)$ into small pieces we can see that for small ϵ, $\int_\gamma \frac{d\xi}{\xi - z} = \int_{\partial\Delta(z;\epsilon)} \frac{d\xi}{\xi - z}$. Using the parametrization $\xi = z + \epsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$ we see that this last integral equals $2\pi i$. □

Page 207, Theorem 10.16, Power Series

Theorem 5.2. *For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by a power series in Ω.*

Proof. Let $z_0 \in \Omega$, $\overline{\Delta(z_0, r)} \subset \Omega$. Let $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$.

By Theorem 10.15 page 207, if $z \in \Delta(z_0; r)$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} d\xi \\
&= \frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z_0} \frac{f(\xi)}{1 - \frac{z - z_0}{\xi - z_0}} d\xi \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi\right)(z - z_0)^n \\
&= \sum_{n=0}^{\infty} c_n (z - z_0)^n;\ c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi
\end{aligned}$$

□

Corollary 5.3. *If $f \in H(\Omega)$, then $f' \in H(\Omega)$.*

Note the proof of the corollary uses Theorem 10.16 but also Theorem 10.6!!

Exercise 5.4. ..

6. Morera's Theorem

Page 208, Morera's Theorem, Theorem 10.17

Theorem 6.1. *Suppose that $f(z)$ is a continuous complex function in an open set Ω such that $\int_{\partial\Delta} f(z)dz = 0$ for every closed triangle in Ω. Then $f \in H(\Omega)$.*

Proof. Start by repeating the proof of Theorem 10.14 as needed here: We follow the proof in Rudin, page 207. Let V be a convex subset of Ω. Fix $a \in V$. Since V is convex, V contains the straight line interval from a to z for every $z \in V$, so we can define

$$F(z) = \int_{[a,z]} f(\xi) d\xi; \ (z \in \Omega).$$

For any z and $z_0 \in V$, the triangle with vertices at a, z_0 and z lies in V. Hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by the hypothesis that the integral around the boundary of triangles are zero.

Fixing z_0, we thus obtain if $z \neq z_0$:

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\xi) - f(z_0)) d\xi.$$

Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)| < \epsilon$ if $|\xi - z_0| < \delta$. Hence the absolute value of the left side is less than ϵ as soon as $|z - z_0| < \delta$. This proves that $F' = f$. In particular, $F \in H(V)$. By the Corollary to Theorem 10.16 page 207, it follows that $f \in H(V)$ and hence in $H(\Omega)$.

□

Exercise 6.2. ..

7. Cauchy Estimates

Cauchy Estimates are page 213, Theorem 10.26

Theorem 7.1. *If $f \in H(D(a;R))$ and $|f(z)| \leq M$ for all $z \in D(a;R)$, then*

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n} (n = 1, 2, 3, \dots).$$

Proof. We use the notation of the proof of Theorem 10.16 page 207.

Let $0 < r < R$, $\gamma = \partial\Delta(a;r)$ and $z \in \Delta(a;r)$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

Differentiating under the integral sign gives

$$f^n(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi.$$

From this the estimate follows □

Page 214. Theorem 10.28

Theorem 7.2. *Suppose $f_j \in H(\Omega), j = 1, 2, 3, \dots,$ and $f_j \to f$ uniformly on compact subsets of Ω. Then $f \in H(\Omega)$, and $f_j' \to f'$ uniformly on compact subsets of Ω.*

Proof. We follow the proof in Rudin page 214. Since the convergence is uniform on each compact disc in Ω, f is continuous. Let Δ be a triangle in Ω. Then Δ is compact, so

$$\int_{\partial\Delta} f(z)dz = \lim_{j\to\infty} \int_{\partial\Delta} f_j(z)dz = 0,$$

by Cauchy's Theorem. Hence Morera's Theorem implies that $f \in H(\Omega)$.

Let K be compact, $K \subset \Omega$. There exists an $r > 0$ such that the union E of the closed discs $\overline{D}(z;r)$, for all $z \in K$, is a compact subset of Ω. Applying Theorem 10.26 page 213 to $f - f_j$, we have

$$|f'(z) - f_j'(z)| \leq r^{-1}\|f - f_j\|_E;\ (z \in K)$$

where $\|f\|_E$ denotes the supremum of $|f|$ on E. Since $f_j \to f$ uniformly on E, it follows that $f_j' \to f'$ uniformly on K. □

Exercise 7.3. ..

8. Simply connected topological space

10.38 page 222. First 10 lines gives a clear definition. probably can just be copied...

Suppose γ_0 and γ_1 are closed curves in a topological space X, both with parameter interval $I = [0,1]$. We say that γ_0 and γ_1 are X- homotopic if there is a continuous mapping H of the unit square $I^2 = I \times I$ into X such that

$$H(s,0) = \gamma_0(s), \quad H(s,1) = \gamma_1(s), \quad H(0,t) = H(1,t) \tag{8.1}$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s,t)$. Then (8.1) defines a one-parameter family of closed curves γ_t in X, which connects γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is X-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is null-homotopic in X. If X is connected and if every closed curve in X is null-homotopic, X is said to be simply connected.

Exercise 8.1. ..

9. Rouche's Theorem

Theorem 9.1. *Suppose that $f \in H(\Omega)$ and $\overline{\Delta}(a,r) \subset \Omega$ and that f has no zeroes on the boundary of $\overline{\Delta}(a,r)$.*
a) Then the number of zeros of f counted with multiplicity, $N(f)$, inside the disc is given by the formula:

$$N(f) = \frac{1}{2\pi i} \int_{\partial\Delta(a,r)} \frac{f'(z)}{f(z)} dz$$

b) If also $g \in H(\Omega)$ and $|g(z) - f(z)| < |f(z)|$ for all $z \in \partial\Delta(a,r)$, then $N(g) = N(f)$.

Proof. We cover a disc with squares so that there is at most one zero of f in each square and piece of disc and none on the boundaries. Then we see, using Cauchy's theorem in a convex set (Theorem 10.26 page 206) that the integral is just the sum over the boundaries of the squares with zeros inside. We further shrink these to small discs centered at the zeros. On such a disc $\overline{\Delta}(z_0, \epsilon)$ we can write $f(z) = (z - z_0)^k g(z)$ with $g(z) \neq 0$ and $k \geq 1$. Then

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{k(z-z_0)^{k-1}g}{(z-z_0)^k g} + \frac{(z-z_0)^k g'(z)}{(z-z_0)^k g(z)} \\ &= \frac{k}{z-z_0} + \frac{g'}{g}. \end{aligned}$$

Then a) follows.

For part b) recall that the integrals vary continuously with f and are always integers. So, let $g_t = tg + (1-t)f, 0 \leq t \leq 1$. Then $|g_t - f| = t|g - f| < |f|$ on $\partial\Delta(a; r)$. Hence g_t has no zeros on the boundary of the disc. Hence we can apply a) to find $N(g_t)$ for each t. Moreover these are integers that vary continuously. Hence

$$N(g) = N(g_1) = N(g_0) = N(f).$$

□

This belongs to item 10.28 page 214:

Theorem 9.2. *a) If Ω is a region and $f \in H(\Omega)$, then $f(\Omega)$ is either a region or a single point.*
b) If f is one-to-one in a disc containing z_0, then $f'(z_0) \neq 0$.

Proof. Let $z_0 \in \Omega$ and let $r > 0, \overline{\Delta}(z_0, r) \subset \Omega$

Then by Theorem 10.16 page 207 we can write $f = \sum_{n=1}^{\infty} c_n (z - z_0)^n$. If all $c_n = 0, n > 0$, then f is constant on the disc. Suppose not. Then there is a smallest $k > 0$ so that $f = z^k g(z)$ where $g(z_0) \neq 0$. We can compare f with $g = z^k g(z_0)$. Then by Rouche's theorem, then for each w close to z_0, there are k preimages counted with multiplicity. In particular f is an open mapping. This proves a). To prove b) note also that if f is one-to-one, then k is one, and hence $f'(z_0) \neq 0$.

□

Next, Theorem 10.24 page 212: The Maximum Modulus Theorem.

Theorem 9.3. *Suppose that Ω is a region, $f \in H(\Omega)$, and $\overline{D}(a; r) \subset \Omega$. Then if $z \in D(a; r)$, then $|f(z)| \leq \max_\theta |f(a + re^{i\theta})|$*

Proof. If not, there is a $b \in \mathbb{D}(a,r)$, $|f(b)| = \max_{\overline{D}(a,r)} |f(z)| > \max_\theta |f(a + re^{i\theta})|$. Let $K = \{z \in \overline{D}(a,r); |f(z)| = |f(b)|\}$. Next choose $c \in K$ with maximum modulus. Then f cannot be an open mapping at c which is a contradiction. □

Exercise 9.4. ..

10. The Schwarz Lemma

Let $f \in H(\Omega)$. We define $\|f\|_\infty = \|f\|_{\Omega,\infty} = \sup_{z\in\Omega} |f(z)|$ and set $H^\infty = H^\infty(\Omega) = \{f \in H(\Omega); \|f\|_\infty < \infty.\}$ Let U denote the open unit disc.

Theorem 10.1. *Theorem 12.2, page 254. The Schwarz Lemma. Suppose* $f \in H^\infty$, $\|f\|_\infty \leq 1$, $f(0) = 0$. *Then*

- *(1)* $|f(z)| \leq |z| (z \in U)$
- *(2)* $|f'(0)| \leq 1$.

If equality holds in (1) for one $z \in U \setminus \{0\}$, *or if equality holds in (2), then* $f(z) = \lambda z$, *where* λ *is a constant,* $|\lambda| = 1$.

Proof. We follow the proof in Rudin. Since $f(0) = 0$, the function $f(z)/z$ has a removable singularity at the origin. So we can write $f(z) = zg(z)$. If $z \in U$, and $|z| < r < 1$ then by the maximum modulus theorem,

$$|g(z)| \leq \max_\theta \frac{|f(re^{i\theta})|}{r} \leq \frac{1}{r}$$

Letting $r \to 1$, we see that $|g(z)| \leq 1$ at every $z \in U$. This gives (1). Since $f'(0) = g(0)$, (2) follows. If $|g(z)| = 1$ for some $z \in U$, then g is constant by the open mapping theorem theorem.

□

Definition 10.2. Definition 12.3. Page 254. Here U means the unit disc, and T means the boundary of the unit disc.

For any $\alpha \in U$, define $\phi_\alpha(z) = \frac{z-\alpha}{1-z\overline{\alpha}}$.

Theorem 10.3. *Theorem 12.4, page 254. Fix* $\alpha \in U$. *Then* ϕ_α *is a one-to-one mapping which carries* T *onto* T, U *onto* U, *and* α *to* 0. *The inverse of* ϕ_α *is* $\phi_{-\alpha}$. *We have* $\phi'_\alpha(0) = 1 - |\alpha|^2$, $\phi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.

Proof. The function ϕ_α is holomorphic everywhere except at the point $1/\overline{\alpha}$ which is a point outside the closed disc. Similarly the function $\phi_{-\alpha}$ is holomorphic in the plane except at the point $-1/\overline{\alpha}$, also outside the unit disc.

Let e^{it} be a point on the unit circle. Then

$$\begin{aligned} |\phi_\alpha(e^{it})| &= \left| \frac{e^{it}-\alpha}{1-e^{it}\overline{\alpha}} \right| \\ &= \left| \frac{e^{it}-\alpha}{e^{-it}-\overline{\alpha}} \right| \\ &= \left| \frac{e^{it}-\alpha}{\overline{e^{it}-\alpha}} \right| \\ &= 1 \end{aligned}$$

This shows that ϕ_α maps T to itself. The same is true for $\phi_{-\alpha}$. Hence, by the maximum modulus principle, both ϕ_α and $\phi_{-\alpha}$ map the open unit disc to itself. Next we calculate

$$\begin{aligned} \phi_{-\alpha} \circ \phi_\alpha(z) &= \frac{\phi_\alpha(z) - (-\alpha)}{1 - \phi_\alpha(z)(-\overline{\alpha})} \\ &= \frac{\frac{z-\alpha}{1-z\overline{\alpha}} - (-\alpha)}{1 - \frac{z-\alpha}{1-z\overline{\alpha}}(-\overline{\alpha})} \\ &= \frac{\frac{z-\alpha}{1-z\overline{\alpha}} + \alpha}{1 + \frac{z-\alpha}{1-z\overline{\alpha}}\overline{\alpha}} \\ &= \frac{(z-\alpha) + \alpha(1-z\overline{\alpha})}{1 - z\overline{\alpha} + z\overline{\alpha} - \alpha\overline{\alpha}} \\ &= \frac{z - z\alpha\overline{\alpha}}{1-\alpha\overline{\alpha}} \\ &= z \end{aligned}$$

Let p, q be points in the closed unit disc. Then $\phi_\alpha(p)$ and $\phi_\alpha(q)$ are both in the closed unit disc. If $\phi_\alpha(p) = \phi_\alpha(q)$, then by applying $\phi_{-\alpha}$ we see that $p = q$. So ϕ_α is one-to-one. If $p \in$ the closed disc, then $\phi_\alpha(\phi_{-\alpha}(p)) = p$, so ϕ_α is also onto.

$\phi_\alpha(\alpha) = 0$. Differentiating we get

$$\begin{aligned} \phi'_\alpha(z) &= \frac{(1-z\overline{z}) - (-\overline{\alpha})(z-\alpha)}{(1-z\overline{\alpha})^2} \\ &= \frac{1-\alpha\overline{\alpha}}{(1-z\overline{\alpha})^2} \end{aligned}$$

Plugging in $z = 0$ and $z = \alpha$, this completes the proof.

$\square$

Exercise 10.4. ..

11. Simply connected regions.

Here and in the proof of the Riemann mapping theorem one needs to use that $\sqrt{f}$ is an analytic function if $f \neq 0$. So we need to prove this. We can first show the inverse function theorem see Theorem 10.30 page 215. And then use that $\sqrt{z}$ is the inverse of $z \to z^2$ and then we compose the analytic function $\sqrt{z}$ and f.

Theorem 11.1. *Theorem 13.11 page 274, points (b) and (j). For a plane region Ω, (b) implies (j):*
(b) Ω is simply connected
(j) If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, then there is a $\phi \in H(\Omega)$ such that $f = \phi^2$.

Proof. Suppose that Ω is simply connected. Let f be nonzero. Pick a point $p \in \Omega$. Let g be a branch of the square root of f in a neighborhood of p. Let q be any point in Ω and let γ be a path from p to q. Using MONODROMY we follow g along γ to get a definition of g at q. We need to show that this value is independent of γ. Suppose γ_1 is another such path. If the two values disagree, then this means that we get the two different square roots of g at the endpoints. Using instead the curve $\gamma_1^{-1} \circ \gamma$ we get a closed loop where we get the two different square roots at p. By simple connectivity we can deform the curves down to a point. Since g had two different values, this remains by continuity true as we shrink the curves down to the point p. But this is impossible. □

Exercise 11.2. ..

12. Normal families

Definition 12.1. (Normal Families, Definition 14.5, Page 281 Suppose $\mathcal{F} \subset H(\Omega)$ for some region Ω. We call $\mathcal{F}$ a normal family if every sequence of members of $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω. The limit function is not required to belong to $H(\Omega)$.

Theorem 12.2. *(Theorem 14.6, Page 282) Suppose $\mathcal{F} \subset H(\Omega)$ and $\mathcal{F}$ is uniformly bounded on each compact subset of the region Ω. Then $\mathcal{F}$ is a normal family.*

Proof. We first need to find a sequence of compact sets $K_n \subset \Omega$ so that $\Omega = \cup_n K_n$ and $K_n \subset K_{n+1}^\circ$. For each n let $K_n = \{z; |z| \leq n, \text{dist}(z, \partial\Omega) \geq 1/n$. Let C_n be the max of $|f|; f \in \mathcal{F}$ on K_n. Pick a sequence f_n. Let p be in Ω. The sequence $f_n(p)$ is bounded. Hence we can choose a subsequence so that $f_{n_k}(p)$ converges to a number which we call $f_\infty(p)$. We can repeat thsi process and find a subsequence which converges for each point say with rational coordinates. We show that this subsequence converges uniformly on each K_n to a limit f_∞ which is then holomorphic on Ω. Note that if $z \in K_n$, and $f \in \mathcal{F}$, then $f(z) = \int_{|\xi - z| = 1/(2n)} \frac{f(\xi) d\xi}{\xi - z}$ and this formula holds also for z' in the disc $\Delta(z, 1/(4n))$. This shows that for such z, z' we have $|f(z) -$

$f(z')| \leq C'_n|z - z'|$ for a fixed constant C'_n. This shows that the sequence $f_n(p)$ is a Cauchy sequence for each $p \in \Omega$. Hence the sequence converges at each point in Ω. Also by uniform continuity, the limit is holomorphic on Ω. □

Exercise 12.3. ..

13. The Riemann mapping theorem

Chapter 14: Conformal mappings. Page 282. Conformal equivalence.

Definition 13.1. We call two regions Ω_1 and Ω_2 conformally equivalent if there exists a $\phi \in H(\Omega_1)$ such that ϕ is one-to-one on Ω_1 and such that $\Omega_2 = \phi(\Omega_1)$, i.e. if there exists a holomorphic one-to-one mapping of Ω_1 onto Ω_2.

Sec. 14.7 The Riemann mapping Theorem. Page 283:

Theorem 13.2. *Every simply connected region Ω in the plane (other than the plane itself) is conformally equivalent to the open unit disc.*

Proof. Let Ω be a simply connected region which is not the whole plane. Pick $z_0 \in \Omega$. Let Σ denote the holomorphic functions on Ω which are 1-1 maps into the unit disc U. Let Σ_0 denote those functions f in Σ for which $f(z_0) = 0$. We want to show that some $\psi \in \Sigma_0$ is onto U.

We first prove that Σ_0 is nonempty.

Let w_0 be a point outside Ω. Since Ω is simply connected, there exists a function $\phi \in H(\Omega)$ so that $\phi^2 = z - w_0$. We show that ϕ is 1-1. Suppose that $\phi(z_1) = \phi(z_2)$. Then $\phi(z_1)^2 = \phi(z_2)^2$. Hence $z_1 - w_0 = z_2 - w_0$ so $z_1 = z_2$. We show also that if $z_1 \neq z_2$ when $\phi(z_1) \neq -\phi(z_2)$: Suppose that $\phi(z_1) = -\phi(z_2)$. Then $\phi(z_1)^2 = \phi(z_2)^2$, so $z_1 = z_2$, a contradiction. Since ϕ is an open mapping the image $\phi(\Omega)$ must contain a disc $\Delta(a, r)$. Hence $\phi(\Omega)$ contains no points in $\Delta(-a, r)$. Hence $|\phi(z) + a| \geq r$ on Ω. It follows that the funtion $\psi = r/(3(\phi + r))$ is one to one and strictly less than 1/2 on Ω. It follows that the function $\psi(z) - \psi(z_0)$ in in Σ_0, so Σ_0 is nonempty.

We show next that if $\psi \in \Sigma_0$ and ψ is not onto, then there is another $\psi_1 \in \Sigma_0$ so that

$$|\psi_1'(z_0)| > |\psi'(z_0)|$$

Recall the functions $\phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ for $|\alpha| < 1$. We recall from Theorem 12.4 page 254 that ϕ_α is one-to-one and onto from the U to U and $\phi_{-\alpha} = \phi_\alpha^{-1}$. Let $\alpha \in U \setminus \phi(\Omega)$. Then $\phi_\alpha \circ \psi \in \Sigma$ and has no zeroes in Ω. Hence there exists a $g \in H(\Omega)$ so that $g^2 = \phi_\alpha \circ \psi$. For the same reason as above, g is one to one on Ω, so is in Σ. Let $\beta = g(z_0)$. Let $\psi_1 = \phi_\beta \circ g$. Then $\psi_1 \in \Sigma$.

Moreover $\psi_1(z_0) = 0$ so $\psi_1 \in \Sigma_0$. Let $s(w) = w^2$. We get

$$\begin{aligned}\psi &= \phi_{-\alpha} \circ g^2 \\ &= \phi_{-\alpha} \circ s \circ g \\ &= \phi_{-\alpha} \circ s \circ \phi_{-\beta} \circ \psi_1 \\ &= (\phi_{-\alpha} \circ s \circ \phi_{-\beta}) \circ \psi_1 \\ &= F \circ \psi_1;\ F = \phi_{-\alpha} \circ s \circ \phi_{-\beta}\end{aligned}$$

Since $\psi_1(z_0) = \psi(z_0) = 0$ we have that $F(0) = 0$. Morover F is not one-to-one. Hence by the Schwarz lemma $|F'(0)| < 1$. We also know that $\psi'(z_0) \neq 0$ since ψ_1 is one-to-one. By the chain rule, $\psi'(z_0) = F'(0)\psi_1'(z_0)$ so we get that $|\psi_1'(z_0)| > |\psi'(z_0)|$ as we claimed.

Let $\eta = \sup\{|\psi'(z_0)| \psi \in \Sigma_0\}$. We have shown that if there exists an $h \in \Sigma_0$ so that $|h'(z_0)| = \eta$, then this h is onto the unit disc. So we show the existence of h.

Since $|\psi| < 1$ on Ω for all $\psi \in \Sigma_0$, Theorem 14.6 page 282 shows that Σ_0 is a normal family. By the definition of η there exists a sequence $\psi_n \in \Sigma_0$ so that $|\psi_n'(z_0)| \to \eta$. By normality, there exists a subsequence which we still call ψ_n which converges uniformly on compact sets to a limit $h \in H(\Omega)$. By Theorem 10.28 page 214, $|h'(z_0)| = \eta$. Since $\psi_n(\Omega) \subset U$, $h(\Omega) \subset \overline{U}$. But $\eta > 0$ so h is an open mapping. Hence $h(\Omega) \subset U$. We only need to show that h is one-to-one. Fix distinct points z_1, z_2 in Ω. Let $\alpha = h(z_1)$ and $\alpha_n = \psi_n(z_1)$. Let $\overline{D}$ be a closed disc with center z_2 so that $h - \alpha$ has no zero on the boundary of $\overline{D}$ and $z_1 \notin \overline{D}$. The functions $\psi_n - \alpha_n$ converge to $h - \alpha$ uniformly on $\overline{D}$. Since the ψ_n are one-to-one, they have no zeros on $\overline{D}$. By Rouche's theorem $h - \alpha$ has no zero on $\overline{D}$. Hence $h(z_1) \neq h(z_2)$. Thus $h \in \Sigma_0$. □

Exercise 13.3. ..

14. Explain what follows

The previous sections are self-contained leading directly to the Riemann mapping theorem. We will start again here, adding what is needed for the remaining sections. So have a clean cut, so that what is before is a complete course and a geodesic to the Riemann map.

Exercise 14.1. ..

15. Koebe 1/4

This is Theorem 14.14 b) on page 288.

Theorem 15.1. *Let $f = z + \sum_{n=2}^{\infty} a_n z^n$ convergent in the unit disc $\Delta(0,1)$. If f is 1-1, then $f(\Delta(0,1)) \supset \Delta(0,1/4)$.*

The functions in this class are said to be in the class $\mathcal{S}$. Definition 14.10 Page 285.

Definition 15.2. $\mathcal{S}$ is the class of all $f \in H(U)$ which are one-to-one in U and which satisfy: $f(0) = 0$ and $f'(0) = 1$.

The proof first proves that $|a_2| \leq 2$. This depends on Theorem 14.12 b) page 285 and the Corollary to Theorem 14.13 page 286. The proof of Theorem 14.13 uses the Cauchy-Riemann equation page?? and Theorem 7.26??

Theorem 15.3. *(Theorem 14.12 page 285) a) If $f \in \mathcal{S}$, $|\alpha| = 1$, and $g(z) = \overline{\alpha} f(\alpha z)$, then $g \in \mathcal{S}$.*
b) If $f \in \mathcal{S}$ there exists $g \in \mathcal{S}$ such that

$$g^2(z) = f(z^2).$$

Theorem 14.13 page 286.

Theorem 15.4. *(Area Theorem) If $F \in H(U \setminus \{0\})$, F is one-to-one in U,and $F = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n$, then $\sum_{n=1}^{\infty} n|\alpha_n|^2 \leq 1$.*

Corollary page 287.

Corollary 15.5. *Under the same hypothesis, $|\alpha_1| \leq 1$.*

We need to do an analysis of what we need in the proof.

Exercise 15.6. ..

16. Continuity at the boundary

Definition 16.1. Simple boundary point. Definition 14.16 on page 289.

This is theorem 14.19 page 290.

Theorem 16.2. *Let Ω be a bounded simply connected domain and suppose that all boundary points are simple. Then the Riemann map from Ω to U extends to a homemorphisms from $\overline{\Omega}$ to $\overline{U}$.*

The proof depends on Theorem 14.18 page 289. We analyze the proof and what it depends on. We need to prove Theorem 14.18 without the Fatou Lemma. Perhaps use reflection. And criteria for holomorphicity using triangles.To get a contradiction...

Exercise 16.3. ..

17. Conformal Mapping between Annuli

Theorem 14.22 page 292.

Theorem 17.1. *Let* $A_i = \{0 < r_i < |z| < R_i < \infty\}$*, i=1,2, be two annuli. Suppose they are biholomorphic. Then* $R_1/r_1 = R_2/r_2$*. Moreover the biholomorphism is either on the form* az *or* a/z *for some nonzero constant* a*.*

Analyze proof. We do it a little differently. We choose α so that $|Fz^\alpha|$ is constant on the boundary and then by max principle is constant inside (since Fe^α is locally holomorphic)...

Exercise 17.2. ..

References

[R] Rudin, Walter: Real and Complex Analysis, Third Edition. McGraw Hill International Editions (1987)